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Real Divergent Nets

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Abstract

Generalization of sequences namely nets were defined and studied. Let (X, T) be a topological space. Let D be a directed set. A function f: D→ X where D is a directed set is called a net. Convergence of nets is defined and properties studied by many mathematicians. Let (X, T) be a topological spaces and f: D→ X where $f(\lambda)=x_\lambda$ be a net in X. Let $x_0 \in X$. We say that (x_λ) converges to $x_0$if every neighbourhood U of x_0, $\lambda_0 \epsilon D$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

In this paper, Divergent Nets in R, Cluster Points and bounded nets are defined and Properties studied.

1. Introduction

Sequence is an important tool in metric spaces. Many properties of metric spaces are studied using sequences. In case of topological spaces, Sequences become inadequate. Hence a generalization of sequences namely nets were defined and studied. Convergence of nets is defined and properties studied by many mathematicians. In case of the Metric space R, we have the concept of divergent sequences. In this paper this concept is extended to nets in R.

2. Divergent Nets in R

Definition 2.1: Let D be a directed set. Let f: D → R be a net in R. f is said to diverge to +∞ if for each k > 0, there exists $\lambda_0 \epsilon D$ such that $f(\lambda) > k$ for all $\lambda \geq \lambda_0$. We write $f \to +\infty$

Example 2.2: Take D=Z. Define f: D → R as $f(\lambda)= |\lambda|$. We shall show that f diverges to +∞. Take any k > 0, choose λ_0, a positive integer in Z where $\lambda_0 > k$.

Now$\lambda \geq \lambda_0 \Longrightarrow \lambda \geq k \Longrightarrow |\lambda| > k \Longrightarrow f(\lambda) > k$

Hence for any k > 0, there exits $\lambda_0 \in D$ such that $f(\lambda) > k$ for all $\lambda \geq \lambda_0$.

Therefore, f diverges to +∞

Definition 2.3: Let D be a directed set. Let f: D → R be a net in R. f is said to diverge to -∞ if for each k < 0, there exits $\lambda_0 \in D$ such that $f(\lambda) < k$ for all $\lambda \geq \lambda_0$. We write $f \to -\infty$

Example 2.4: Take D = Z. Define f: D → Z as $f(\lambda)= - |\lambda|$.

We shall prove that f diverges to -∞

Take any k < 0, choose $\lambda_{0,}$ a positive integer in Z where $\lambda_0 > -k$. Take any $\lambda \geq \lambda_{0.}$ Clearly λ is a positive integer.

Now $\lambda \geq \lambda_0$ and $\lambda_0 > -k$. Hence, > - k. Since λ is positive $f(\lambda) = |\lambda|$. Since k< 0,

we have -k > 0. Now $|\lambda| > -k$. $-|\lambda| < k$. Hence $f(\lambda) < k$. Hence, we have $\lambda_0 \in D$ such that $f(\lambda) < k$ for all $\lambda \geq \lambda_0$. Therefore, f diverges to -∞

Definition 2.5: Let D be a directed set. Let f: D → R be a net in R. f is said to be divergent if either f diverges to +∞ or f diverges to -∞.

Example 2.6: Take D=Z. Define f: D → R as $f(\lambda) = \lambda^2$ Clearly f diverges to +∞. Hence f is divergent.

Example 2.7: Take D = N. Define f: D → R as $f(\lambda) = -\lambda$. Clearly f diverges to -∞. Hence f is divergent.

Result 2.8: We know that every sequence in R is a net in R.

Let f be a sequence in R. f: N → R is a sequence. At the same time N is a directed set. Hence f: N → R is a Net. We can consider f as a sequence and as a net.

If f as a sequence diverges to +∞ then f as a net diverges to +∞.

If f as a sequence diverges to -∞ then f as a net diverges to -∞.

If f as a sequence is divergent then f as a net is divergent. Hence,we have the following result.

Result 2.9: The concept of divergent net in R is an extension of the concept of divergent sequences in R.

3. Cluster Points ∞ and -∞

Definition 3.1: Let f: D → R be a net in R. ∞ is said to be a cluster point of f if for each k>0 and for each $\lambda_0 \in D$, there exists $\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) > k$.

Example 3.2: Take D=Z. Define f: D → R as $f(\lambda) = \lambda$. We shall show that ∞ is a cluster point off

Take any k>0. Choose $\lambda_{0,}$ a positive integer such that $\lambda_0 > k$. Choose a positive integer λsuch that $\lambda > \lambda_0$.

Now $\lambda > \lambda_0 \Longrightarrow \lambda > k \Longrightarrow f(\lambda) > k$. Hence ∞ is a cluster point.

Definition 3.3: Let f: D → R be a net in R. $-\infty$ is said to be a cluster point of fif for each k<0 and for each $\lambda_0 \in D$, there exists $\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) < k$.

Example 3.4: Take D=Z. Define f: D → R as $f(\lambda) = -\lambda$. Clearly $-\infty$ is a cluster point of f

Theorem 3.5: Let f: D → R be a net in R. If f diverges to $+\infty$, then $+\infty$ is a cluster point of f

Proof: Let f: D → R be a net in R. f diverges to $+\infty$. We shall prove that ∞ is a cluster point of f

Take any k > 0. Take any $\lambda_0 \in D$. Since f diverges to $+\infty$ for the given k > 0, there exists $\lambda_1 \in D$ such that $f(\lambda) > k$ for all $\lambda \geq \lambda_1$. Consider $\lambda_{0,1} \in D$. Since D is a directed set, there exist $\lambda_2 \in D$ such that $\lambda_2 \geq \lambda_0$ and $\lambda_2 \geq \lambda_1$. Since $\lambda_2 \geq \lambda_1$ we have $f(\lambda_2) > k$. Hence for each k > 0 and for each $\lambda_0 \in D$, we have $\lambda_2 \in D$ such that$\lambda_2 \geq \lambda_0$ and $f(\lambda_2) > k$. Hence ∞ is a cluster point off

Result 3.6: Converse is not true.

If ∞ is a cluster point of f then f need not diverges to ∞.

Example 3.7: Take D=N. Define f: D → R as $f(\lambda) = \begin{cases} \lambda \text{ if } \lambda \text{ is odd} \\ -\lambda \text{ if } \lambda \text{ is even} \end{cases}$ We shall show that ∞ is a cluster point.

Take any k>0 Take any $\lambda_0 \in D$. Then λ_0 is a natural number. Choose a natural number λsuch that $\lambda > \lambda_0$ and $\lambda > k$ and λ is odd. Now $f(\lambda) = \lambda > k$. Now for k>0 for$\lambda_0 \in D$, we have $\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) > k$. Hence ∞ is a cluster point of f

We shall show that f does not diverge to ∞.

Suppose not, if f diverges to ∞. Take any k >0. Since f diverges to ∞, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0 \Longrightarrow f(\lambda) > k$. Now let λ be an even number such that $\lambda \geq \lambda_0$. Now $f(\lambda) = -\lambda < k$. $\Longrightarrow\Longleftarrow$ hence f does not diverge to ∞.

Theorem 3.8: Let f: D → R be a net in R. If f diverges to $-\infty$, then $-\infty$ is a cluster point of f

Proof: f: D → R be a net in R. f diverges to $-\infty$, we shall prove that,$-\infty$ is a cluster pointof fTake any k<0 and take $\lambda_0 \in D$. Since f diverges to $-\infty$, for this k<0, there exists $\lambda_1 \in D$ such that $\lambda \geq \lambda_1 \Longrightarrow f(\lambda) < k$. Now $\lambda_{0,1} \in D$ and D is a directed set. Hence there exists $\lambda_2 \in D$ such that $\lambda_2 \geq \lambda_0$and $\lambda_2 \geq \lambda_1$. Now $\lambda_2 \geq \lambda_1 \Longrightarrow f(\lambda_2) < k$. Hence, we have $\lambda_2 \geq \lambda_0$and $f(\lambda_2) < k$. Hence $-\infty$ is a cluster point of f

Result 3.9: Converse is not true. If $-\infty$ is a cluster point of f then f need not diverge to $-\infty$.

Example 3.10: Take D=N. Define f: D → R as $f(\lambda) = \begin{cases} \lambda \text{ if } \lambda \text{ is odd} \\ -\lambda \text{ if } \lambda \text{ is even} \end{cases}$ We shall show that $-\infty$ is a cluster point.

Take any $k < 0$ Take any $\lambda_0 \in D$. λ_0 is a natural number. Choose a natural number $\lambda \in D$ such that $\lambda \geq \lambda_0$ and $\lambda > -k$ and λ is even number. Now $\lambda > -k \Longrightarrow -\lambda < k \Longrightarrow f(\lambda) < k$. Hence for given $k>0$ and $\lambda_0 \in D$, there exists,$\in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) < k$. Hence $-\infty$ is a cluster point of f

We shall show that f does not diverge to $-\infty$.

Suppose not, if f diverges to $-\infty$. Take any $k < 0$. Since f diverges to $-\infty$, there exists $\lambda_0 \in D$ such that $\lambda \geq \lambda_0 \Longrightarrow f(\lambda) < k$. Let λ be an odd number such that $\lambda \geq \lambda_0$. Now $f(\lambda) = \lambda > k$

$\Longrightarrow\Longleftarrow$Hence f does not diverge to $-\infty$.

Theorem 3.11: Let $f: D \rightarrow R$ be a net in R. If ∞ and $-\infty$ are cluster points off then f is not divergent.

Proof: $f: D \rightarrow R$ is a net in R. ∞ and $-\infty$ are cluster points of f

Claim 1: f does not diverge to$+\infty$

Suppose not, if f diverges to $+\infty$. Take any $k>0$. Since f diverges to $+\infty$, there exists$\lambda_0 \in D$ such that $\lambda \geq \lambda_0 \Longrightarrow f(\lambda) > k \Longrightarrow f(\lambda) > 0$

Consider $-k < 0$ and $\lambda_0 \in D$. Since $-\infty$ is a cluster point, there exists$\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) < -k$. $\Longrightarrow f(\lambda) < 0$. We have $f(\lambda) > 0$ and $f(\lambda) < 0$ $\Longrightarrow\Longleftarrow$ Hence f does not diverge to $+\infty$

Claim 2:f does not diverge to $-\infty$

Suppose not, if f diverges to $-\infty$. Take any $k < 0$. Since f diverges to $-\infty$, there exists$\lambda_0 \in D$ such that $\lambda \geq \lambda_0 \Longrightarrow f(\lambda) < k \Longrightarrow f(\lambda) < 0$

Consider $-k > 0$ and $\lambda_0 \in D$. Since ∞ is a cluster point, there exists$\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) > -k$. $\Longrightarrow f(\lambda) > 0$. We have $f(\lambda) < 0$ and $f(\lambda) > 0$ $\Longrightarrow\Longleftarrow$ Hence f does not diverge to $-\infty$

from claim 1 and 2 we have f is not divergent.

Result 3.12: Converse is not true.

f is not divergent does not imply that ∞ and $-\infty$ are cluster points.

Example 3.13: Take D =N. Define $f: D \rightarrow R$ as $f(\lambda) = 1$ for all $\lambda \in D$. Clearly f is not divergent and ∞ and $-\infty$ are not cluster points.

4. Bounded Nets:

Definition 4.1: Let $f: D \rightarrow R$ be a net in R. f is said to be bounded above if there exists $k \in R$ such that $f(\lambda) \leq k$, for all $\lambda \in D$.

Example 4.2: Take D = Z. Define $f: D \rightarrow R$ as $f(\lambda) = 1$ for all $\lambda \in D$. Clearly for $k = 1$, $f(\lambda) \leq k$, for all $\lambda \in D$. Hence f is bounded above.

Definition 4.3: Let $f: D \rightarrow R$ be a net in R. f is said to be bounded below if there exists$L \in R$ such that $f(\lambda) \geq L$, for all $\lambda \in D$.

Example 4.4: Take D = Z. Define $f: D \rightarrow R$ as $f(\lambda) = |\lambda|$ for all $\lambda \in D$. Now we have $f(\lambda) \geq 0$, for all $\lambda \in D$. Hence f is bounded below.

Definition 4.5: Let $f: D \rightarrow R$ be a net in R. f is said to be bounded if f is both bounded above and bounded below

Example 4.6: Take D=Z. Define f: D → R as f(λ) = $\begin{cases} -1 \text{ if } \lambda \text{ is odd} \\ 1 \text{ if } \lambda \text{ is even} \end{cases}$

We have f(λ) ≤ 1, for all λ∈ D. Hence f is bounded above.

We have f(λ) ≥ -l, for all λ∈ D. Hence f is bounded below. Hence f is bounded.

Result 4.7: f is bounded if and only if there exists k > 0 such that |f(λ)| ≤ k, for all λ∈ D.

Theorem 4.8: If f is bounded above then f does not diverge to +∞.

Proof: f: D → R is bounded above, there exists k ∈ R such that f(λ) ≤ k for all λ∈ D. we can assume k > 0 for this k >0, there exists no λ_0 such that f(λ) ≥ k for all $\lambda \geq \lambda_0$. Hence f does not diverge to +∞.

Result 4.9: Converse is not true.

f does not diverge to ∞ ⇏f is bounded above

Example 4.10: Take D=N Define f:D → R as f(λ) = $\begin{cases} \lambda \text{ if } \lambda \text{ is odd} \\ 1 \text{ if } \lambda \text{ is even} \end{cases}$

f does not diverge to +∞. At the same time f is not bounded above.

Theorem 4.11: If f: D → R is bounded below then f does not diverge to -∞.

Proof: f: D → R is bounded below. Hence there exists L ∈ R such that f(λ) ≥ L for all λ∈ D. we can assume L < 0. Now for this L < 0, there does not exist λ_0∈ D satisfying f(λ) < L for all $\lambda \geq \lambda_0$. Hence f does not diverge to -∞.

Result 4.12: Converse is not true. f does not diverge to -∞⇏ f is bounded below

Example 4.13: Take D=N. Define f: D → R as f(λ) = $\begin{cases} -\lambda \text{ if } \lambda \text{ is odd} \\ 0 \text{ if } \lambda \text{ is even} \end{cases}$

Here f does not diverge to -∞. At the same time f is not bounded below.

Theorem 4.14: If f: D → R is bounded then f is not divergent.

Proof: f: D → R is bounded. Hence f is bounded above and f is bounded below.

By theorem 4.8, f does not diverge to +∞

By theorem 4.11, f does not diverge to -∞

Hence f is not divergent.

Result 4.15: Converse is not true.

f is not divergent ⇏f is bounded

Example 4.16: Take D=N. Define f: D → R as f(λ) = $\begin{cases} \lambda \text{ if } \lambda \text{ is odd} \\ -\lambda \text{ if } \lambda \text{ is even} \end{cases}$

Clearly f is not divergent. Also, f is not bounded.

Theorem 4.17: If f: D → R is bounded above then ∞ is not a cluster point

Proof: f: D → R is bounded above there exists k > 0 such that that f(λ) ≤ k for all λ∈ D. Consider k > o and any λ_0∈ D. For each $\lambda \geq \lambda_{0,}$ f(λ) ≤ kthere exists, no λ with f(λ) > k

Hence,∞ is not a cluster point.

Result 4.18: Converse is not true.

∞ is not a cluster point ⇏f is bounded above.

Example 4.19: Take D=Z. Define f: D → R as $f(\lambda) = \begin{cases} 1 & \text{if } \lambda \geq 0 \\ -\lambda & \text{if } \lambda < 0 \end{cases}$

………5,4,3,2,1,1,1,1,1,……..

Take k =2. $\lambda_0 = 10$. For any $\lambda \geq \lambda_0$, $f(\lambda) = 1$.

There exist $\lambda \geq \lambda_0$ such that $f(\lambda) > k$. Hence ∞ is not a cluster point. It is clear that f is not bounded above.

Theorem 4.20: If f: D → R is bounded below, then -∞ is not a cluster point.

Proof: f: D → R is bounded below. There exists L < 0 such that $f(\lambda) > L$ for all $\lambda \in D$.

Consider L< 0, take any $\lambda_0 \in R$. For any $\lambda \geq \lambda_0$, $f(\lambda) > L$ there exists no $\lambda \in D$ such that $\lambda \geq \lambda_0$ and $f(\lambda) < L$. Hence, -∞ is not a cluster point.

Result 4.21: Converse is not true.

-∞ is not a cluster point $\not\Rightarrow$ f is bounded below.

Example 4.22: Take D=Z Define f: D → R as $f(\lambda) = \begin{cases} 1 & \text{if } \lambda \geq 0 \\ \lambda & \text{if } \lambda < 0 \end{cases}$

………-5,-4,-3,-2,-1,1,1,1,1,……..

Take k=-1. $\lambda_0 = 5$. For any $\lambda \geq \lambda_0$, $f(\lambda) = 1 > -1$.

Hence -∞ is not a cluster point. It is clear that f is not bounded below.

Theorem 4.23: If f: D → R is bounded then ∞ and -∞ are not cluster points.

Proof: f: D → R is bounded. Hence f is bounded above and f is bounded below. By theorem 4.17∞ is not a cluster point. By theorem 4.20, -∞ is not a cluster point.

Result 4.24: Converse is not true.

Example 4.25: Take D=Z Define f: D → R as $f(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0 \\ \lambda & \text{if } \lambda < 0 \end{cases}$

………-5,-4,-3,-2,-1,1,1,1,1,……..

It is clear that ∞ and -∞ are not cluster points. Also, f is not bounded.

References

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